

PARAMETRIC H-PRINCIPLE FOR HOLOMORPHIC IMMERSIONS WITH APPROXIMATION

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ABSTRACT. We prove the parametric homotopy principle for holomorphic immersions of Stein manifolds into Euclidian space and the homotopy principle with approximation on holomorphically convex sets. We write an integration by parts like formula for the solution f to the problem $Lf|_{\Sigma} = g$, where L is a holomorphic vector field, semi-transversal to analytic variety Σ .

1. INTRODUCTION AND RESULTS

A C^1 map $f : X \rightarrow Y$ between smooth manifolds X and Y is an immersion if $\theta = df : TX \rightarrow TY$ is injective. It was shown by Hirsch [7] that given an injective bundle homomorphism $\theta : TX \rightarrow TY$ there exists a continuous homotopy of injective homomorphisms $\theta_t : TX \rightarrow TY$ with $\theta_0 = \theta$ and $\theta_1 = df$ for some immersion $f : X \rightarrow Y$. Following [1] we say that smooth immersions $X \rightarrow Y$ satisfy h-principle.

This result cannot be generalized to holomorphic maps between arbitrary complex manifolds. For example there can be a lot of injective bundle maps $TX \rightarrow T\mathbb{C}^N$, where X is a compact complex manifold, but there are no holomorphic immersions $X \rightarrow \mathbb{C}^N$ when $\dim X \geq 1$, since the only compact analytic sets in \mathbb{C}^N are finite. Additionally, if $X = \mathbb{C}^n$, there are examples of Kobayashi hyperbolic manifolds Y such that every holomorphic map $\mathbb{C}^n \rightarrow Y$ has rank strictly less than n everywhere. However, immersions of Stein manifolds into \mathbb{C}^N satisfy the following holomorphic homotopy principle by Eliashberg-Gromov (§2.1.5, [1]):

*Suppose that the cotangent bundle T^*X of a Stein manifold X with $\dim X = n$ is generated by $q > n$ holomorphic $(1, 0)$ -forms $\varphi_1, \dots, \varphi_q$.*

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Then q -tuple $\varphi = (\varphi_1, \dots, \varphi_q)$ can be changed (through q -tuples generating T^*X) to (df_1, \dots, df_q) where $(f_1, \dots, f_q) : X \rightarrow \mathbb{C}^q$ is a holomorphic immersion.

The sketch of the proof is included in [1], however a lot of technical details are missing. The main idea described in [1] is using a solution to the problem $Lf|_\Sigma = g$ to replace forms one by one by differentials. Suppose φ_q is being replaced by a differential. Then $\Sigma \subset X$ would be an analytic set where forms $\varphi_1, \dots, \varphi_{q-1}$ fail to span T^*X , $L : X \rightarrow TX$ would be a holomorphic vector field with $\varphi_1 L = 0, \dots, \varphi_{q-1} L = 0$ on Σ , $g = \varphi_q L$ a holomorphic function and f an unknown holomorphic function. Let $\varphi_{q,t} = (1-t)\varphi_q + tdf$. The desired homotopy through q -tuples spanning T^*X would be then $(\varphi_1, \dots, \varphi_{q-1}, \varphi_{q,t})$ since $\varphi_{q,t} L = \varphi_q L \neq 0$ on Σ .

To solve the above problem the semi-transversality condition $L \pitchfork_{\text{semi}} \Sigma$ (see definition 4.1) introduced in [1] is needed. In section 4 we show in detail how to obtain this condition for \mathcal{L} and Σ' which appear in the proof of lemma A_4'' in [1], p. 69. We use the jet transversality theorem for holomorphic maps to reduce the set of non semi-transversal points by obtaining semi-transversality on the complements of analytic sets and making sure we do not get any new non semi-transversal points (see lemma 4.4).

We additionally prove the homotopy principle with approximation (theorem 1.1) by writing down a formula for the solution f of $Lf|_\Sigma = g$ (theorem 3.3) and observing that given a holomorphically convex compact set $K \subset X$ we get $|f|_K < M|g|_K$, where M depends only on the restriction of data L, Σ on K . Here we additionally allow small perturbations of Σ and L .

We show how the proof of theorem 1.1 can be generalized to families of initial $(1,0)$ -forms and use the formula (see theorem 3.6) to establish the parametric homotopy principle (theorem 1.3). As a corollary of this and the Oka principle for sections of bundles we show (corollary 1.5) that if differentials of two immersions $f_1, f_2 : X \rightarrow \mathbb{C}^q$ are homotopic through q -tuples of linearly independent forms, then immersions f_1 and f_2 are homotopic through immersions.

Theorem 1.1. *Let X be a Stein manifold and let $q > n = \dim X \geq 1$. Let $\varphi = (\varphi_1, \dots, \varphi_q) : X \rightarrow (T^*X)^q$ be $(1,0)$ -forms on Stein manifold X which are holomorphic on a compact, holomorphically convex set $K \subset X$. Let $\epsilon > 0$. Suppose that $\text{rank } \varphi(x) = n$ for all $x \in X$. Let $g = (g_1, \dots, g_q)$ be functions, holomorphic on K such that $\varphi = dg$ on K .*

Then there exists a continuous homotopy $H : X \times [0, 1] \rightarrow (T^*X)^q$ such that

- (i) for each $t \in [0, 1]$ are $(1, 0)$ -forms $H(\cdot, t)$ holomorphic on K ,
 $H(\cdot, 0) = \varphi$ and $H(\cdot, 1) = df$ for some holomorphic functions
 $f = (f_1, \dots, f_q) : X \rightarrow \mathbb{C}^q$ satisfying $|f - g|_K < \epsilon$,
- (ii) $\text{rank } H(x, t) = n$ for all $(x, t) \in X \times [0, 1]$.

Remark 1.2. Case $n = 1$.

When X is 1-dimensional Stein manifold (open Riemann surface), the set $\Sigma = \{x \in X : \text{rank}(\varphi_1, \dots, \varphi_{q-1}) < 1\}$ is a discrete set. By [4], Theorem 2.1, the problem $Lf|_\Sigma = g$ is solvable. Therefore immersions $X \rightarrow \mathbb{C}^q$ satisfy homotopy principle for all $q \geq 1$ (case $n = q = 1$ is described in [4], Theorem 2.5).

Case when $\dim X = q > 1$ is still an open problem; method described in [1] fails since one form is removed and the remaining forms should generate T^*X somewhere. However immersions $X \rightarrow \mathbb{C}^n$, where $X \subset \mathbb{C}^n$ is a contractible domain, satisfy homotopy principle and the obtained homotopy class consists of one element:

Suppose $X = \mathbb{C}^n$. Let $\varphi = (\varphi_1, \dots, \varphi_n) : \mathbb{C}^n \rightarrow (T^*\mathbb{C}^n)^n$ be holomorphic $(1, 0)$ -forms with $\text{rank } \varphi = n$. There is a homotopy from φ to $df = (dz_1, \dots, dz_{n-1}, dz_n)$, satisfying (ii) in the main theorem:

Let $A = (a_j^k)_{j,k} : \mathbb{C}^n \rightarrow \mathbb{C}^{n^2}$ be a (unique) holomorphic map such that $\sum_{j=1}^n a_j^k \varphi_j = dz_j$ for $j = 1, \dots, n$. Let $G : \mathbb{C}^n \times [0, 1] \rightarrow GL_n(\mathbb{C}) \subset \mathbb{C}^{n^2}$ be a homotopy of holomorphic maps from identity to A . A continuous homotopy exists since X is contractible and $GL_n(\mathbb{C})$ connected. By Oka principle [6] for sections of the bundle $(X \times \mathbb{C}^{n^2}) \setminus \Sigma \rightarrow X$, where fiber $\mathbb{C}^{n^2} \setminus \Sigma_x = GL_n(\mathbb{C})$ is homogeneous (hence it satisfies CAP), there exists a homotopy $H' : X \times [0, 1] \rightarrow GL_n(\mathbb{C})$ of holomorphic sections from A to identity. Set $H(x, t) = H'(x, t) \cdot \varphi(x)$.

Question. How to construct a homotopy when $X = \mathbb{C}^n \setminus \{0\}$?

Theorem 1.3 (parametric homotopy principle). *Suppose $\varphi : X \times \Omega \rightarrow (T^*X)^q$ is a holomorphic map with $\text{rank } \varphi(x, s) = n$ on $X \times \Omega$, where the parameter space Ω is a Stein manifold. Then there exists a continuous map $H : X \times \Omega \times [0, 1] \rightarrow (T^*X)^q$, holomorphic in $(x, s) \in X \times \Omega$ with $H(x, s, 0) = \varphi(x, s)$, $H(x, s, 1) = df_s(x)$, and $H(\cdot, s, \cdot)$ satisfies (ii) in theorem 1.1 for all $s \in \Omega$.*

Remark 1.4. For applications the existence of continuous homotopy $\varphi : X \times \Omega \rightarrow T^*X$ satisfying the rank condition (ii) in theorem 1.1 is

usually sufficient since it implies the existence of holomorphic homotopy satisfying the same condition with additional interpolation and approximation of given φ :

Homotopy is a section of the bundle $\pi \times \text{id} : (T^*X)^q \setminus A \times \Omega \rightarrow X \times \Omega$, where A is an analytic variety describing q -tuples of forms with $\text{rank} < n$. Fiber of the bundle is the complement of algebraic variety $\{\lambda \in \mathbb{C}^{nq} : \text{rank } \lambda < n\}$ and has codimension $q - n + 1 \geq 2$, hence Oka principle for sections (theorem 1.1 in [6]) applies.

Corollary 1.5. *Let $G : X \times [0, 1] \rightarrow (T^*X)^q$ be a continuous homotopy of holomorphic forms satisfying (ii) in theorem 1.1 and $G(\cdot, 0) = df_1$, $G(\cdot, 1) = df_2$, where $f_1, f_2 : X \rightarrow \mathbb{C}^q$ are holomorphic immersions. Then there exists a continuous homotopy $F : X \times [0, 1] \rightarrow \mathbb{C}^q$ of holomorphic immersions such that $F(\cdot, 0) = f_1$, $F(\cdot, 1) = f_2$.*

Proof. We can construct continuous homotopy $G' : X \times \mathbb{C} \rightarrow (T^*X)^q$, holomorphic near $\{0\} \cup \{1\}$, such that $G'(\cdot, 0) = G(\cdot, 0)$, $G'(\cdot, 1) = G(\cdot, 1)$ and $\text{rank } G' = n$ (define homotopy as a constant on $\{\text{Re } t \in [-1/4, 1/4]\} \cup \{\text{Re } t \in [3/4, 5/4]\} \subset \mathbb{C}$). Note that $G' \times \text{id}$ with $\text{rank } G' = n$ is a section of holomorphic bundle $\pi \times \text{id} : (T^*X)^q \setminus A \times \mathbb{C} \rightarrow X \times \mathbb{C}$ (see the above remark). By theorem 1.1 in [6] there exists a holomorphic section G'' such that $G''|_{\{0,1\} \times X} = G'$.

Now apply theorem 1.3 with $\varphi = G''$ to obtain $H : X \times \mathbb{C} \times [0, 1] \rightarrow (T^*X)^q$. The path consisting of the straight lines from $(s, t) = (0, 0)$ to $(0, 1)$, then from $(0, 1)$ to $(1, 1)$ and lastly from $(1, 1)$ to $(1, 0)$, describes the homotopy F . Note that all the perturbations in the proof of theorems 1.1 and 1.3 are exact. \square

2. PROOF OF MAIN THEOREM

Proof of Theorem 1.1: Final homotopy H will be a conjunction of homotopies from steps (1) and (2).

(1) Approximation with globally defined holomorphic forms.

Using Oka principle (Lemma 2.1) construct a homotopy H_0 satisfying (ii) such that $H_0(\cdot, 0) = \varphi$ and $H_0(\cdot, 1)$ is holomorphic on X with $|H_0(\cdot, 1) - \varphi|_K < \epsilon_1$. Here ϵ_1 is chosen small enough (more precisely $(M+1)^q \epsilon_1 < \delta$ and $(M+1)^q \epsilon_1 < \epsilon$ where $M = M(g|_K)$ and $\delta = \delta(g|_K)$ are from theorem 3.3; see also step (2)) such that the approximations in step (2) will be small enough. Note that exactness of the forms on K is not preserved in this step.

(2) **Replacing forms one by one with differentials.** For each $k = 1, \dots, q$ we inductively construct homotopies H_k satisfying (ii) such

that forms in $H_k(\cdot, 1)$ $M\epsilon_1(1+M)^{k-1}$ -approximate forms in $H_{k-1}(\cdot, 1) = H_k(\cdot, 0)$ on K , where $H_0(\cdot, 1) = \varphi$. Since $\epsilon_1(1+M)^{k-1} + M\epsilon_1(1+M)^{k-1} = \epsilon_1(1+M)^k$, forms in $H_k(\cdot, 1)$ will $\epsilon_1(1+M)^k$ -approximate forms in original φ . Each homotopy H_k will replace k -th form by a differential of holomorphic function.

-(2.1) **Obtaining semi-transversality.** Suppose that at the beginning of k -th step we have the forms

$$(\varphi_1 = dh_1, \dots, \varphi_{k-1} = dh_{k-1}, \varphi_k, \dots, \varphi_q) = H_{k-1}(\cdot, 1).$$

First we use the homotopy H'_k to generically perturb forms φ_j , $j \neq k$ in order to obtain semi-transversality condition $L \pitchfork_{\text{semi}} \Sigma$ (lemmas 4.3, 4.4 and proposition 4.5) needed in step (2.2). Here $\Sigma = \Sigma(\varphi) = \{x \in X : \text{rank}(\varphi_j(x))_{j \neq k} < n\}$, more precisely $\Sigma = \bigcap_{\{\alpha_1, \dots, \alpha_n\} \subset \{1, \dots, q\} \setminus \{k\}} \{x \in X : \text{rank}(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_n}) < n\}$. $L : X \rightarrow TX$ is a holomorphic vector field (uniquely defined on Σ since $\text{rank} \varphi = n$), satisfying $\varphi_j L = 0$ for all $j \neq k$ and $\varphi_k L = 1$ on Σ . Globally defined L is obtained by extension using Cartan's theorems.

-(2.2) **Solving $Lf_k|_\Sigma = 1$ with approximation on K .** Find the holomorphic function $f_k = g_k + h : X \rightarrow \mathbb{C}$ where h satisfies $Lh|_\Sigma = 1 - Lg_k|_\Sigma$. Since perturbation of $L|_K$ remains bounded when φ is perturbed a little on K ($L \mapsto L|_\Sigma : \Gamma(X, TX) \rightarrow \Gamma(\Sigma, TX)$ is continuous, linear surjection, hence open by open mapping theorem) and $\Sigma \cap K$ is perturbed a little if $\varphi|_K$ is perturbed a little, $1 - Lg_k = 1 - dg_k L = \varphi_k L - dg_k L$ is close to 0 on K , hence there exists a solution h which is $M|1 - dg_k L|_K$ -close to 0 on K by Theorem 3.3. Note that functions f_j (and \tilde{f}_j) in theorem 3.3 are in our case the defining equations for the set Σ , described in detail in the proof of Lemma 4.3. Hence f_k approximates g_k on K . Define the homotopy as

$$H''_k(\cdot, t) = (\varphi_1, \dots, \varphi_{k-1}, (1-t)\varphi_q + tdf_k, \varphi_{k+1}, \dots, \varphi_q).$$

Note that H''_k satisfies (ii) since $\varphi_q L = df_k L = 1$. H_k is a conjunction of homotopies H'_k and H''_k , that is $H_k(\cdot, t) = H'_k(\cdot, 2t)$ if $0 \leq t \leq 1/2$ and $H_k(\cdot, t) = H''_k(H'_k(\cdot, 1), 2t - 1)$ if $1/2 \leq t \leq 1$. \square

Lemma 2.1. *Given forms φ as in the main theorem 1.1 there is a continuous homotopy $H : X \times [0, 1] \rightarrow (T^*X)^q$ of forms holomorphic on K and satisfying (ii) in main theorem, such that $H(\cdot, 1)$ is holomorphic on X and approximates φ on K .*

Proof. There are finitely many holomorphic forms Φ_1, \dots, Φ_N ($N \geq n$) with rank n on X . Hence we can write $\varphi_k = \sum_{j=1}^N a_j^k \Phi_j$ for some functions a_j^k , holomorphic on K . $(\text{id}, a_j^k) : X \rightarrow X \times \mathbb{C}^{Nq}$ is a section

of the holomorphic fiber bundle $\pi : (X \times \mathbb{C}^{Nq}) \setminus A \rightarrow X$ where $A = \{(x, a) \in X \times \mathbb{C}^{Nq} : \text{rank}(\varphi_1, \dots, \varphi_q) \leq n-1\} \subset X \times \mathbb{C}^{Nq}$ is analytic variety and $\pi : X \times \mathbb{C}^q \rightarrow X$ is a standard projection. Since the complements of fibers $A_x = A \cap (\{x\} \times \mathbb{C}^q)$ are affine hyperplanes of codimension equal to $q - (n-1) \geq 2$, we can apply theorem 1.1(B) in [6] to the sections of the bundle $\pi : (X \times \mathbb{C}^{Nq}) \setminus A \rightarrow X$ and obtain the desired homotopy. \square

Proof of theorem 1.3. We follow the proof of theorem 1.1 and check that all the steps can be done holomorphically in parameter s . Define $\tilde{\Sigma} = \{(x, s) : x \in \Sigma_s\}$, where $\Sigma_s = \{x \in X : \text{rank} \varphi(x, s) < n\}$. We work with family of vector fields $\tilde{L} : X \times \Omega \rightarrow TX$, which is uniquely defined on $\tilde{\Sigma}$; global extension is obtained using Cartan's theorems.

Step (2.1): When obtaining semi-transversality in step (2.1) define

$$\tilde{\Sigma}'' = \{(x, s) \in \tilde{\Sigma} : (L_U)^k h_{j_1, \dots, j_n}^l(x) = 0 \text{ for all } L_U, l, 1 \leq k \leq n+1\}.$$

The goal, to reduce $\tilde{\Sigma}''$ to \emptyset , is achieved as before. Observe that df in lemma 4.3 can be chosen with holomorphic dependence on s . f is a generic map, obtained from the jet transversality theorem, such that the jet of f is transversal to some variety $\Lambda = \Lambda(s)$ in the jet space. Suppose $\Lambda(s)$ is defined by $g(x, s, \lambda_x) = 0$, where $(x, \lambda_x) \in J^{m+1}(X, \mathbb{C})$. Define $\tilde{\Lambda} = \{(x, s, \lambda_x, \lambda_s) \in J^{m+1}(X \times \Omega, \mathbb{C}) : g(x, s, \lambda_x) = 0\}$ and note that $\tilde{\Lambda}$ is an analytic variety with $\text{codim } \tilde{\Lambda} \geq \text{codim } \Lambda$. The jet of generic $F : X \times \Omega \rightarrow \mathbb{C}$ is transversal to $\tilde{\Lambda}$. Hence when $\text{codim } \Sigma(s) = m > \dim(X \times \mathbb{C}) = n+1$, $j_x^{m+1} f_s$ will miss $\Sigma(s)$ for all s . Hence a generic perturbation of forms in step (2.1) can be constructed of the form $\varphi + df_s$, where $f_s = F(\cdot, s)$ depends holomorphically on s .

Step (2.2) To get a solution f_s (holomorphically depending on parameter s) to the problem $L_s f|_{\Sigma_s} = 1$ use theorem 3.6. \square

3. SOLVING $Lf|_{\Sigma} = g$ WITH APPROXIMATION ON COMPACTS AND WITH PARAMETERS

Let X be a Stein manifold, $\Sigma \subset X$ an analytic set and $L : X \rightarrow TX$ a nowhere on Σ vanishing holomorphic vector field. Let $\mathcal{J}(\Sigma)$ be the sheaf of ideals in \mathcal{O}_X , consisting of the germs of holomorphic functions that vanish on Σ .

Example 3.1. Solution f to the problem $Lf|_{\Sigma} = g$ does not always exist. Suppose that (extreme opposite of the assumptions in Theorem 3.3) L is tangential to Σ to infinite order at all points of Σ . Then

$f' = f|_{\Sigma}$ is a solution of the problem $Lf' = g$, where f' is holomorphic on Σ .

The simplest example where there is no solution f is $L = \frac{\partial}{\partial z}$, $g = 1$ and $X = \{(z, w) \in \mathbb{C}^2 : z \neq 0\}$. More complicated example [8] is a domain $\Sigma \in \mathbb{C}^3$, biholomorphic to a polydisc, where the problem $\frac{\partial f'}{\partial z_1} = g$ is not solvable for some holomorphic functions g . In this example there is a complex line $L = \{z_2 = \text{const.}, z_3 = \text{const.}\} \subset \mathbb{C}^3$, such that $L \setminus (L \cap \Sigma)$ has bounded components. Choose a point (z_1^0, z_2, z_3) in such a bounded component and let $g(z)$ be holomorphic extension of $1/(z_1 - z_1^0)$ from $L \cap D$. Also note that a generic function has a nonzero residue, hence $Lf = g$ is not solvable for most functions g . The problem is not solvable even if Σ is Runge domain in \mathbb{C}^n [8].

But in the case when there exists a solution f' , the solution to $Lf|_{\Sigma} = g$ is any holomorphic extension f of f' , since $L(f - f') = 0$ on Σ (tangentiality).

Question 3.2. 1. Can we solve the problem $Lf|_{\Sigma} = 1$ if we know how to solve the problem $Lf|_{\Sigma} \neq 0$?
2. Can we solve the problem $Lf \neq 0$ on a sphere $S \subset \mathbb{C}^n$ such that f approximates given g on a compact $K \subset \mathbb{C}^n$ provided that $Lg \neq 0$ on K ?

The following theorem shows that the solution f to the problem $Lf|_{\Sigma} = g$ changes a little on a compact K if the input data L, g are changed a little on K .

Theorem 3.3. *Let $K \subset X$ be a compact, holomorphically convex set. Suppose there exist holomorphic functions $f_1, \dots, f_N \in \mathcal{J}(\Sigma)$ such that*

$$(3.1) \quad \{f_1 = 0, \dots, L^N f_1 = 0, \dots, f_N = 0, \dots, L^N f_N = 0\} = \emptyset.$$

There exists $\delta > 0$ and $M = M(K, f|_K, L|_K) > 0$, such that given analytic set $\tilde{\Sigma} \subset X$, holomorphic vector field \tilde{L} with $|\tilde{L} - L|_K < \delta$, holomorphic functions $\tilde{f}_j \in \mathcal{J}(\tilde{\Sigma})$ satisfying $\sum_{j=1}^N |\tilde{f}_j - f_j|_K < \delta$ and (3.1) and a holomorphic function $\tilde{g} : X \rightarrow \mathbb{C}$, we have $\tilde{L}\tilde{f}|_{\tilde{\Sigma}} = \tilde{g}|_{\tilde{\Sigma}}$ for some holomorphic function $\tilde{f} : X \rightarrow \mathbb{C}$ satisfying $|\tilde{f}|_K \leq M|\tilde{g}|_K$.

Proof. Condition (3.1) implies the existence of holomorphic functions $a_j^1, \dots, a_j^N : X \rightarrow \mathbb{C}$ ($j = 1, \dots, N$) such that

$$(3.2) \quad \sum_{j=1}^N a_j^1 L f_j + \dots + \sum_{j=1}^N a_j^N L^N f_j = 1$$

on Σ , hence $\sum_{j=1}^N ga_j^1 Lf_j + \dots + \sum_{j=1}^N ga_j^N L^N f_j = g$. The solution to the problem $Lf|_\Sigma = g$ is then

$$(3.3) \quad \begin{aligned} f = & \sum_{j=1}^N ga_j^1 f_j + \dots + \sum_{j=1}^N ga_j^N L^{N-1} f_j - \\ & - \left(\sum_{j=1}^N L(ga_j^2) f_j + \dots + \sum_{j=1}^N L(ga_j^N) L^{N-2} f_j \right) - \\ & - \dots - \left(\sum_{j=1}^N L^{N-1}(ga_j^N) f_j \right) \end{aligned}$$

The proof is completed by applying the following lemma to the coefficients a_j^k in equation (3.2). \square

Remark 3.4. (1) Since L is semi-transversal to Σ , there are (Lemma 4.2) finitely many holomorphic functions f_1, \dots, f_N with $f_j \in \mathcal{J}(\Sigma)$ satisfying (3.1). Hence the problem $Lf|_\Sigma = g$ has a solution. But in our application we need to show that the sup norm $|f|_K$ of the solution is small whenever $|g|_K$ is small. Moreover, Σ changes during the proof of the main theorem.

(2) Let $\Lambda \subset X$ be an analytic set. If $g \in \mathcal{J}(\Lambda)^N$ then we can conclude $f \in \mathcal{J}(\Lambda)$. Can we obtain such f when $g \in \mathcal{J}(\Lambda)$?

Lemma 3.5. *Let $K \subset X$ be a compact, holomorphically convex sets and let $f = (f_1, \dots, f_N) : X \rightarrow \mathbb{C}^N$ be holomorphic functions without common zero and let $a = (a_1, \dots, a_N) : X \rightarrow \mathbb{C}^N$ be holomorphic functions such that $a \cdot f = \sum_{j=1}^N a_j f_j = 1$. Set $\delta = 1/(2|a|_K) > 0$.*

For every $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_N)$ without common zeroes such that $|\tilde{f} - f|_K = \sum_{j=1}^N |\tilde{f}_j - f_j|_K < \delta$ there exist functions $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_N) : X \rightarrow \mathbb{C}^N$ with $\tilde{a} \cdot \tilde{f} = 1$ and $|\tilde{a} - a|_K < 4|a|_K^2 |f - \tilde{f}|_K$.

Proof. Let $a_0 = a/a \cdot \tilde{f}$. Now $|a_0 - a|_K \leq 2|a|_K |1 - a \cdot \tilde{f}|_K \leq 2|a|_K^2 |f - \tilde{f}|_K$. Let $a' : X \rightarrow \mathbb{C}$ be such that $a' \cdot \tilde{f} = 1$. Since $a_0 - a' : K \rightarrow X$ is a section of coherent analytic sheaf $\mathcal{F} = \{a \in \mathcal{O}_X^N : a \cdot \tilde{f} = 0\}$, we can approximate it on K with a global section $a'' \in \Gamma(X, \mathcal{F})$ such that $|a_0 - a' - a''|_K < 2|a|_K^2 |f - \tilde{f}|_K$. Set $\tilde{a} = a' + a''$. \square

The proof of Theorem 3.3 also gives the following parametric version:

Theorem 3.6. *Let $\Omega \subset \mathbb{C}$ be a Stein manifold. Let $G : X \times \Omega \rightarrow \mathbb{C}$ be a holomorphic function and let $\tilde{\Sigma} \subset X \times \Omega$ be an analytic set. Let*

$L : X \times \Omega \rightarrow TX$ be a holomorphic map such that $L_s = L(\cdot, s) : X \rightarrow TX$ is a holomorphic vector field for all $s \in \Omega$.

Set $\Sigma_s = \{x \in X : (x, s) \in \tilde{\Sigma}\}$ and $g_s = G(\cdot, s)$. Suppose that $L_s \pitchfork_{\text{semi}} \Sigma_s$ for all $s \in \Omega$. Then there is a holomorphic function $F : X \times \Omega \rightarrow \mathbb{C}$, such that $f_s = F(\cdot, s)$ is a solution to the problem $Lf|_{\Sigma_s} = g_s|_{\Sigma}$ for all $s \in \Omega$.

Proof. Use the (proof of) lemma 4.2 to choose a number $N \in \mathbb{N}$ and holomorphic functions $F_1, \dots, F_N : X \times \Omega \rightarrow \mathbb{C}$ in $\mathcal{J}(\tilde{\Sigma})$ such that $\{(x, s) : F_1 = 0, L_s^N F_1 = 0, \dots, L_s^N F_N = 0\} = \emptyset$. Now choose holomorphic functions $A_j^l : X \times \Omega \rightarrow \mathbb{C}$ as in (3.2) and write the solution F as in the proof of theorem 3.3. \square

4. SEMI-TRANSVERSALITY

Definition 4.1. Let $L : X \rightarrow TX$ be a holomorphic vector field on a complex manifold X of dimension n and let $\Sigma \subset X$ be an analytic set. L is *tangent to Σ with order less or equal k* at $x \in \Sigma$, if there is a holomorphic function $f \in \mathcal{J}(\Sigma) = \{g \in \mathcal{O}_X : g|_{\Sigma} = 0\}$ with $L_x^j f = L(\dots L(f) \dots)_x \neq 0$ for some $j \leq k$. We say that L is *semi-transversal* (see [1]) to analytic variety $\Sigma \subset X$ and write $L \pitchfork_{\text{semi}} \Sigma$, if L is tangent to Σ with finite order at each point of Σ .

The following lemma explains why it is sufficient to check semi-transversality condition for local extensions of vector field $L|_{\Sigma}$.

Lemma 4.2. Suppose that each $x \in \Sigma$ has a neighborhood U and a holomorphic vector field $L_U : U \rightarrow TX$ with $(L_U)|_{\Sigma \cap U} = L|_{\Sigma \cap U}$, such that L_U is tangent to Σ with order less or equal N at each point of $U \cap \Sigma$. Then there is a finite number $N' \leq n + 1$ of functions $f_1, \dots, f_{N'} \in \mathcal{J}(\Sigma)$ such that $\{f_1 = 0, \dots, L^N f_1 = 0, \dots, f_{N'} = 0, \dots, L^N f_{N'} = 0\} = \emptyset$. In other words, L is tangent to Σ with the order less than N at each point of Σ .

Proof. Also see lemma (§2.1.5, p.66) in [1]. Inductively define a sequence of sheafs of ideals by $I_0 = \mathcal{J}(\Sigma)$ and $I_{k+1} = I_k + \mathcal{O}_X \cdot \{Lf : f \in I_k\}$ for $k \geq 1$. Sheafs I_k satisfy $I_k \subseteq I_{k+1}$ for all k . Note that replacing L by L' such that $L - L' = 0$ on $\Sigma \cap U$ does not change ideal I_{k+1} over U since $\mathcal{J}(\Sigma) \subset I_k$. Hence the ideals are dependent only of $L|_{\Sigma}$. Suppose L_U is tangent to Σ with order less or equal k on $U \cap \Sigma$. Then $I_k = \mathcal{O}_X$ over U . This implies the existence of $N' \in \mathbb{N}$ and functions $f_1, \dots, f_{N'}$:

Choose $x \in \Sigma$ and a function $f_1 \in \mathcal{J}(\Sigma)$ with $L_x^{N_1} f_1 \neq 0$ for some $N_1 \leq N$. Such function exists because $I_{N_1} = \mathcal{O}_X$ on a neighborhood of x . Therefore $\dim\{f_1 = 0, Lf_1 = 0, \dots, L^{N_1} f_1 = 0\} < n = \dim X$. Suppose we have constructed functions $f_1, \dots, f_k \in \mathcal{J}(\Sigma)$ and numbers $N_1 \leq \dots \leq N_k \in \mathbb{N}$ such that the dimension of analytic set $\Sigma_1^k = \{f_1 = 0, \dots, L^{N_k} f_k = 0\}$ is less or equal $n - k$. We can assume that $\Sigma_1^k \subset \Sigma$; if not just add defining functions for Σ . Choose a point x_j in each $(n - k)$ -dimensional irreducible component of Σ_1^k . Proceed as in the proof of Proposition (§5.7) in [3], which proves that analytic sets in complex manifolds can be defined by finitely many equations, by selecting the function $f_{k+1} \in \mathcal{J}(\Sigma)$ such that $x_j \notin \Sigma_1^{k+1} = \{f_1 = 0, \dots, L^{N_{k+1}} f_{k+1} = 0\}$ for all j . Such function is of the form $f_{k+1} = \sum_{l \in \mathbb{N}} c_l f_{k+1}^l$, where $L^{N_l} f_{k+1}^l(x_l) \neq 0$ for some $N_l \leq N$ and all $l \in \mathbb{N}$. Functions $f_{k+1}^l \in \mathcal{J}(\Sigma)$ exist since $I_{N_l} = \mathcal{O}_X$ on a neighborhood of x_l . Numbers c_l are chosen inductively on l such that $L^{N_l} f_{k+1}(x_l) \neq 0$ for all $l \in \mathbb{N}$ and such that the sum converges uniformly on compacts in X . Hence $\dim \Sigma_1^{k+1} < \dim \Sigma_1^k$. Therefore in $N' \leq n + 1$ many steps we get $\Sigma_1^{N'} = \emptyset$. Set $N = N_{N'}$.

□

Now we describe how to obtain semi-transversality condition $L \pitchfork_{\text{semi}} \Sigma$ needed in the step (2.2) of the proof of the main theorem. All notation is as in step (2) of the proof. We will work with $k = q$. Let

$$\Sigma = \cap_{1 \leq j_1 < \dots < j_n < q} \Sigma(\varphi_{j_1}, \dots, \varphi_{j_n})$$

where $\Sigma(\varphi_{j_1}, \dots, \varphi_{j_n}) = \{x \in X : \text{rank}(\varphi_{j_1}, \dots, \varphi_{j_n}) < n\}$. Let h_{j_1, \dots, j_n}^l be the defining equations for each set of the intersection. Let $L : X \rightarrow TX$ be an extension (existence follows from Cartan theorems) of holomorphic field $\Sigma \rightarrow TX$, uniquely defined on Σ by $\varphi_1 L = 0, \dots, \varphi_{q-1} L = 0, \varphi_q L = 1$.

Local extensions L_U of vector field $L|_\Sigma$:

$\text{rank}(\varphi_1, \dots, \varphi_q) = n$ on X , hence for each $x \in \Sigma$ there are $n - 1$ $(\varphi_2, \dots, \varphi_n$ for sake of notation) forms among $\varphi_1, \dots, \varphi_{q-1}$ which together with φ_q span T^*V on a neighborhood $U = V \setminus \Sigma(\varphi_2, \dots, \varphi_n, \varphi_q)$ of v . Every choice of such $(n - 1)$ -tuple of forms defines the extension $L_U = L(\varphi_2, \dots, \varphi_n) : U \rightarrow TX$ of the vector field $L|_\Sigma$ defined by $\varphi_1(L|_\Sigma) = 0, \dots, \varphi_{q-1}(L|_\Sigma) = 0$ and $\varphi_q(L|_\Sigma) = 1$. There are at most $\binom{q-1}{n-1}$ possible choices. Let Σ'' be defined as a finite intersection

$$\Sigma'' = \{x \in \Sigma : (L_U)^k h_{j_1, \dots, j_n}^l(x) = 0 \text{ for all } L_U, l, 1 \leq k \leq n + 1\},$$

In other words L_U is tangent to Σ with the order less than n at all points $v \in U \cap (\Sigma \setminus \Sigma'')$. If $\Sigma'' = \emptyset$ then for each $x \in \Sigma$ there is a local extension L_U tangent with finite order to Σ on U , $x \in U$. Lemma 4.2 then shows that $L \pitchfork_{\text{semi}} \Sigma$.

We will reduce Σ'' to \emptyset in countably many steps. First we show how to establish upper bound on the order of tangentiality outside some analytic set Σ' .

Lemma 4.3. *Choose $x \in \Sigma$, a compact $K \subset X$, $k \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $x \notin \Sigma' = \Sigma(\varphi_2, \dots, \varphi_n, \varphi_q)$ (for each $x \in \Sigma$ there are $n-1$ forms that span T^*V together with φ_q). Set $U = X \setminus \Sigma'$.*

Then there exists a holomorphic function $f \in \mathcal{J}(\Sigma')^k$ such that the vector field $L_U = L(\varphi_1 + df, \varphi_2, \dots, \varphi_{q-1}, \varphi_q)$ is tangential to $\Sigma(\varphi_1 + df, \varphi_2, \dots, \varphi_{q-1})$ with the order less than $n+1$ on $\Sigma \setminus \Sigma'$ and df is ϵ -close to 0 on K .

Proof. Such f will be obtained from jet-transversality theorem for holomorphic functions on a Stein manifold. Note that $(\cap_{j=1}^{q-1} \ker \varphi_j)|_{\Sigma \setminus \Sigma'} = (\cap_{m=2}^n \ker \varphi_m)|_{\Sigma \setminus \Sigma'}$ is independent of φ_1 . Hence by changing φ_1 we do not change L on $\Sigma \setminus \Sigma'$. Let U be a neighborhood of an arbitrarily choosen point in $X \setminus \Sigma'$, such that in the local coordinates on complex manifold X we have $L = \frac{\partial}{\partial x_1}$.

The defining equation for $\Sigma(\varphi_1 + df, \varphi_2, \dots, \varphi_n)$ on a neighborhood U of point x is $\det[\varphi_1 + df, \varphi_2, \dots, \varphi_n] = 0$. By expanding the determinant of $n \times n$ matrix

$$A = [\varphi_1 + df, \dots, \varphi_n]$$

by the first row we get

$$(4.1) \quad h(f, v) = \det A = \sum_{j=1}^n \left(\frac{df}{dx_j} + \varphi_1^j \right)(x) A_j(x) = 0,$$

where A_j is $(1, j)$ -minor (depending only on $\varphi_2, \dots, \varphi_n$). The tangentiality condition in Lemma 4.3 will be satisfied if for $m = n + 1$ the vector field $\frac{\partial}{\partial x_1}$ is tangential to Σ with the order at most m on U . We need the function f such that

$$Lh = \frac{\partial h(f, x)}{\partial x_1} = 0 \dots, L^m h = \frac{\partial^m h(f, x)}{\partial x_1^m} = 0$$

is true nowhere on $X \setminus \Sigma'$. This is equivalent to $j^{m+1} f(X \setminus \Sigma') \cap \Lambda = \emptyset$, where $\Lambda \subset J^{m+1}(X \setminus \Sigma', \mathbb{C})$ is an analytic set. By observing the definition of $h(f, x)$ we see that $\text{codim } \Lambda = m > n$ if $m > n$ (note that at least one minor A_j is nonzero at each point). By jet transversality

theorem for holomorphic functions $X \setminus \Sigma' \rightarrow \mathbb{C}$ the jet of a generic holomorphic function $f : X \setminus \Sigma' \rightarrow \mathbb{C}$ (we can choose f to be close to 0 on K) is transversal to Λ on $V \setminus \Sigma'$. To complete proof choose exhaustion of $X \setminus \Sigma'$ by compacts K_j and construct a sequence of uniformly on compacts in $X \setminus \Sigma'$ converging generic holomorphic maps $f_j \in \mathcal{J}^k(\Sigma)$, such that the jet of each map is transversal to Λ on a compact K_j in $V \setminus \Sigma'$ and f_j approximates f_{j-1} on K_{j-1} . Set $f = \lim_{j \rightarrow \infty} f_j$.

□

This describes the reduction of the set Σ'' of "bad" points in Σ .

Lemma 4.4. *Notation is from the previous lemma. Denote by $\Sigma''(\varphi_1)$ the dependence of Σ'' on φ_1 . There is a holomorphic function $f \in \mathcal{J}(\Sigma')^{n+2}$ with $|f|_K < \epsilon$ such that:*

- (1) $\tilde{L}_U = L(\varphi_1 + df, \varphi_2, \dots, \varphi_{q-1})$ is tangential to Σ with the order less than $n + 1$ on $\Sigma \setminus \Sigma'$,
- (2) $\Sigma''(\varphi_1 + df) = \Sigma''(\varphi_1) \cap \Sigma'$,
- (3) The homotopy $t \mapsto (\varphi_1 + tdf, \varphi_2, \dots, \varphi_{q-1}, \varphi_q)$ satisfies (ii) in theorem 1.1.

Proof. First part follows from $\Sigma \subset \Sigma(\varphi_1 + df, \varphi_2, \dots, \varphi_n)$ and the previous lemma.

The second part follows from $\Sigma''(\varphi_1 + df) \subset \Sigma \cap \Sigma'$ and the following. Suppose $x \in (\Sigma \cap \Sigma') \setminus \Sigma''(\varphi_1)$. By definition of $\Sigma''(\varphi_1)$ the order of tangency of $L|_U$ to Σ near x is less or equal n . The difference between the old and the new defining function for Σ near x lies in $\mathcal{J}(\Sigma')^{n+1}$ since $f \in \mathcal{J}(\Sigma')^{n+2}$. Closer examination of the linear system defining $L|_U$ shows the same is true for the change of $L|_U$. Then the order of tangentiality of $L|_U$ to Σ at x is preserved; this is a simple consequence of the definition of the order of tangency and of the derivation rule for products.

The last part follows from the definition of $\Sigma' = \Sigma(\varphi_2, \dots, \varphi_n, \varphi_q)$ and the fact that $df = 0$ on Σ' . □

Proposition 4.5. *Let $K \subset X$ be a compact set and let $\epsilon > 0$. There is a continuous homotopy $\Psi : X \times [0, 1] \rightarrow (T^*X)^{q-1}$ such that*

- (1) $\Psi(0) = (\varphi_1, \dots, \varphi_{q-1})$,
- (2) for every $x \in \Sigma$ the corresponding vector field $L_U = L(\Psi(1))$ is tangential to Σ with order at most $n + 1$ on a neighborhood U of x
- (3) Homotopy $(\Psi(\cdot, t), \varphi_q)$ satisfies (ii) in theorem 1.1 and $\Psi(t)$ is ϵ -close to $\Psi(0)$ on K for all $t \in [0, 1]$

Proof. Homotopy is obtained by consecutively joining countably many homotopies described by Lemma 4.4. We choose an exhaustion of Stein manifold X with compacts. At each step the forms are modified by homotopy (Lemma 4.4) and we make sure the sequence of modified forms converges uniformly on compacts (lemma 4.4).

The goal is to reduce the "bad" set Σ'' of points, tangential with infinite order, to \emptyset . At each step Σ'' is contained in some analytic variety. We show that at each step the number of irreducible components of highest dimension of this variety is reduced:

Let $\Sigma_1'' = \Sigma'' \subset \Sigma_1 = \Sigma$ and let Σ^1 be the union of all irreducible components of Σ_1 having nonempty intersection with Σ_1'' . We have $\Sigma_1'' \subset \Sigma^1$. Now choose $x \in \Sigma_1''$ lying in highest dimensional irreducible component of Σ^1 . Note that $\Sigma_1' = \Sigma'$ in the proof of lemma 4.3 is chosen such that some neighborhood of x in X is disjoint to Σ_1' . Now Σ_2'' (new Σ'') is equal to $\Sigma_1'' \cap \Sigma_1' \subset \Sigma^1 \cap \Sigma_1'$. But $\Sigma^2 = \Sigma^1 \cap \Sigma_1'$ is missing at least one of highest dimensional irreducible component of Σ^1 . Since analytic variety can have at most countably many highest dimensional irreducible components, we get $\Sigma'' \subset \emptyset$ in countably many steps. \square

Remark 4.6. Countably many steps are needed in the reduction in our proof. That is reason why the final homotopy is only piecewise smooth(countably many pieces), since it is obtained as a conjunction of countably many homotopies.

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